Electron Emission in Intense Electric Fields.

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§ 1. *Introduction.*—The main features of the phenomenon of the extraction of electrons from cold metals by intense electric fields are well known, and an approximate theory of the effect was first developed by Schottky.* More recently the experimental data have been much improved. notably by Millikan and Eyring,† and Millikan and Lauritsenf. The theory has been considered afresh by O. W. Richardson§ and by Houston:: working with Sommerfeld. It seems to us, however, that there is still room for improrement in the theoretical exposition and its correlation with the experiments. Neither O. W. Richardson nor Houston really treat the theory in the simple straightforward way which is now possible in the new mechanics, using the revived electron theory of metals which we owe to Sommerfeld. Again, while Millikan and Lauritsen seem to have established quite definitely the laws of dependence of the emission on the field strength F, they speak of the implications of their result in a way which is hard to justify and might in certain circumstances prove to be definitely misleading.

Millikan and Lauritsen show that a plot of log I, where I is the current, against 1/F yields a good straight line whenever the experimental conditions are sufficiently stable, At ordinary temperatures these currents are completely independent of the temperature. The formula for these currents is

$$I = Ce^{-a/F}, \tag{1}$$

which is, of course, indistinguishable from

$$I = CF^2e^{-a/F}. \tag{2}$$

Millikan and his associates have also shown that as the higher temperatures, at which ordinary thermionic emission begins: are approached: the strong field emission does become sensitive to temperature and finally blends into the thermionic.

* Schottky, ' Z. f. Physik,' vol. 14, p. 80 (1923).
† Millikan and Eyring, ' Phys. Rev.,' vol. 27, p. 51 (1926).
‡ Millikan and Lauritsen, ' Proc. Nat. Ac. Sci.,' vol. 14, p. 45 (1928).
§ O. W. Richardson, ' Roy. Soc. Proc.,' A, vol. 117, p. 719 (1928).
|| Houston, ' Z. f. Physik,' vol. 47, p. 33 (1928).

On the strength of these facts they suggest that perhaps there may exist a general formula for the current

$$I = A\,(T + cF)^2\, e^{-b/(T+cF)}. \qquad (3)$$

valid over wide ranges of temperature (T) and field strength. This formula is, of course, correct for large T and small F and also for large F and small T. At intermediate strengths it does not appear to have been tested quantitatively and therefore yet awaits experimental and theoretical investigation. On one other deduction, however, they lay great emphasis, and this we find liable to mislead. They assert that a distinction should be drawn between the electrons which can function as thermions and the ordinary conduction electrons which yield the emission at great field strengths and are absolutely independent of the temperature.

In this paper, therefore, we extend the results of Nordheim* to include the effect of an external field using the same methods and the same underlying picture of the metal (Sommerfeld's). We establish the formula (2) independent of T at low temperatures in agreement with experiment. We fail to find any theoretical justification for (3), though, of course, some justification may exist. Combining our results with those of Nordheim we show that Sommerfeld's picture of a metal yields the formula both for strong fields and for thermionic emission. A single set of free or conduction electrons distributed according to the Fermi-Dirac statistics suffices for both purposes. It is for this reason that we take exception to the statements of Millikan and Lauritsen recorded above. We gladly admit that everything they say can be reconciled with our theory, but only by a forced interpretation of their statements which we consider lay the emphasis wrongly.

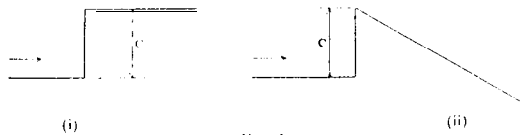
Our calculations are closely allied to some recent work by Oppenheimer† on the hydrogen atom in an external electric field. Oppenheimer notices that his work has a bearing on the emission of electrons in strong fields, but he does not pursue the matter further. For our purposes the calculations can be shorn of irrelevancies and made so much simpler that it is worth while attacking the problem *de novo*.

§ 2. *The Reflection of Electrons at a Potential Jump when an Electric Field acts on one Side.*—In the paper quoted Nordheim has calculated exactly the reflection or emission coefficients for electrons of given energy W incident from the left on a surface at which their potential energy is suddenly increased by

* Nordheim, 'Z. f. Physik,' vol. 46, p. 833 (1928).

† Oppenheimer, 'Phys. Rev.,' vol. 13, p. 66 (1928).

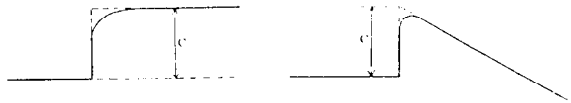
an amount C, fig. 1 (i). The problem is one-dimensional, and W is the kinetic energy of the electron's motion normal to the surface of the metal. The rest



(i) (ii)

FIG. 1.

of its kinetic energy is irrelevant to the problem. He has shown further that the results are only slightly modified by modifying the form of the step, *e.g.*, by slightly rounding off the corners. We shall here calculate in a similar way the emission coefficients for electrons of given energy when a uniform external field acts, so that the potential energy of the electrons is as shown in fig. 1 (ii). The corner at the top will, of course, really be rounded off in both cases by the image effect. This will seriously alter the emission coefficient for non-zero external field for electrons of incident energy nearly equal to W. We shall see, however, that this modification is unimportant in calculating the strong field emission at ordinary temperatures. The potential energies rounded off by the image effect will be somewhat as shown in fig. 2.



(i) (ii)

FIG. 2.

In order to study the emission through the potential energy step of fig. 1 we have only to solve the wave equations

$$\frac{d^2\psi}{dx^2} + \kappa^2 (W - C + Fx)\,\psi = 0 \qquad (x>0), \qquad (4)$$

$$\frac{d^2\psi}{dx^2} + \kappa^2 W \psi = 0 \qquad (x<0), \qquad (5)$$

subject to the conditions that ψ and $d\psi/dx$ are continuous at $x = 0$ and that for $x>0$ ψ represents a stream of electrons progressing to the right only. The constant κ is defined by

$$\kappa^2 = 8\pi^2 m/h^2. \qquad (6)$$

At all ordinary and low temperatures practically all the electrons are so distributed according to Sommerfeld's formula (see equation (19)) that $W < C$, so that we have to discuss the case in which we pass through a zero of the coefficient of ψ.

Write

$$\left(-\frac{C-W}{F}+x\right)(\kappa^2 F)^{\frac{1}{3}} = y$$

then equation (4) becomes

$$\frac{d^2\psi}{dy^2} + y\psi = 0,$$

of which the solutions are expressible in terms of Bessel's functions* of order $\frac{1}{3}$,

$$\psi = \sqrt{y}\, J_{\pm\frac{1}{3}}(\tfrac{2}{3}y^{\frac{3}{2}}). \tag{7}$$

We require that solution which for large x (*i.e.*, y) represents a wave travelling to the right. Therefore

$$\psi = \sqrt{y}\, H_{\frac{1}{3}}^{(2)}(\tfrac{2}{3}y^{\frac{3}{2}}), \tag{8}$$

where $H^{(2)}$ denotes the second function of Hankel. For large y

$$\psi \backsim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{y^{\frac{1}{2}}} \frac{1}{(2/3)^{\frac{1}{6}}} e^{-i\left[\frac{2}{3}y^{\frac{3}{2}} - \frac{5\pi}{12}\right]}. \tag{9}$$

Observe that as $y \longrightarrow \infty$

$$|\psi|^2 \backsim \frac{A}{(W-C+Fx)^{\frac{1}{2}}} = \frac{A'}{v},$$

where A, A′ are constants and v is the velocity of the electrons. Hence the density of the electron stream behaves as it should.

For $x < 0$ we take

$$\psi = \frac{1}{W^{\frac{1}{4}}}\left[ae^{-ix\sqrt{W}} + a'e^{\frac{1}{4}ix\sqrt{W}}\right]. \tag{10}$$

We have now to study what happens to $H_{\frac{1}{3}}^{(2)}$ as we pass back to $x = 0$ from large x through the zero of y. There is no real ambiguity, for ψ is represented by a power series of integral powers of y. In order, however, to use the normal asymptotic expansion for $H_{\frac{1}{3}}^{(2)}(\frac{2}{3}y^{\frac{3}{2}})$, which is valid in the range

$$-2\pi < \arg(y^{\frac{3}{2}}) < \pi,$$

we must take for negative y

$$y = -y' = y'e^{-\pi i},$$

* For all formulæ used here involving Bessel's functions, see Watson, "Bessel's Functions." They are easily found so that it is unnecessary to give detailed references.

and this convention we shall use throughout. At $x = +0$ we then find

$$\psi(0) = \left(\frac{C-W}{F}\right)^{\frac{1}{3}} e^{-\frac{1}{3}\pi i}(\kappa^2 F)^{\frac{1}{3}}\, H_{\frac{1}{3}}^{(2)}\left(e^{-\frac{1}{2}\pi i}\tfrac{2}{3}\kappa\sqrt{F}\left(\frac{C-W}{F}\right)\right), \tag{11}$$

$$\left(\frac{d\psi}{dx}\right)_0 = \tfrac{1}{2}\left(\frac{C-W}{F}\right)^{-\frac{1}{3}} e^{\frac{1}{3}\pi i}(\kappa^2 F)^{\frac{1}{3}}\, H_{\frac{1}{3}}^{(2)}\left(e^{-\frac{1}{2}\pi i}\tfrac{2}{3}\kappa\sqrt{F}\left(\frac{C-W}{F}\right)\right)$$
$$-\left(\frac{C-W}{F}\right)^{\frac{1}{3}}(\kappa^2 F)^{\frac{1}{3}}\kappa(C-W)^{\frac{1}{2}}\, H_{\frac{1}{3}}^{(2)\prime}\left(e^{-\frac{1}{2}\pi i}\tfrac{2}{3}\kappa\sqrt{F}\left(\frac{C-W}{F}\right)\right). \tag{12}$$

These values are to be equated to the values of ψ and $\partial\psi/\partial x$ for $x = -0$ derived from (10).

Since the only quantity of physical interest is $|a|^2/|a'|^2$, which determines the emission coefficient, we can simplify the resulting equations by omitting any common factor real or complex which is thereby absorbed in both a and a'. We shall write also

$$\tfrac{2}{3}\kappa\sqrt{F}\left(\frac{C-W}{F}\right)^{\frac{1}{2}} = Q,$$

so that Q is real and in practice large. Further,

$$H_{\frac{1}{3}}^{(2)\prime}(e^{-\frac{1}{2}\pi i}Q) = e^{\frac{1}{2}\pi i}\frac{dH_{\frac{1}{3}}^{(2)}}{dQ}(e^{-\frac{1}{2}\pi i}Q).$$

The equations of continuity of ψ and $d\psi/dx$ can therefore be reduced to

$$a + a' = W^{\frac{1}{4}}\left(\frac{C-W}{F}\right)^{\frac{1}{3}} H_{\frac{1}{3}}^{(2)}(e^{-\frac{1}{2}\pi i}Q), \tag{13}$$

$$-a + a' = \frac{i}{\kappa W^{\frac{1}{4}}}\left[\tfrac{1}{2}\left(\frac{C-W}{F}\right)^{-\frac{1}{3}} H_{\frac{1}{3}}^{(2)}(e^{-\frac{1}{2}\pi i}Q) + \frac{C-W}{F}\kappa\sqrt{F}\frac{dH_{\frac{1}{3}}^{(2)}}{dQ}(e^{-\frac{1}{2}\pi i}Q)\right]. \tag{14}$$

By the definition of the functions of Hankel we can express $H_{\frac{1}{3}}^{(2)}(e^{-\frac{1}{2}\pi i}Q)$ in terms of the real functions $I_{\pm\frac{1}{3}}(Q)$ by the equation

$$H_{\frac{1}{3}}^{(2)}(e^{-\frac{1}{2}\pi i}Q) = \frac{-1}{\sin\frac{1}{3}\pi}\{I_{-\frac{1}{3}}(Q) + e^{\frac{1}{3}\pi i}I_{\frac{1}{3}}(Q)\}. \tag{15}$$

Let us now write

$$\alpha + i\beta = \frac{I'_{-\frac{1}{3}} + e^{\frac{1}{3}\pi i}I'_{\frac{1}{3}}}{I_{-\frac{1}{3}} + e^{\frac{1}{3}\pi i}I_{\frac{1}{3}}}, \tag{16}$$

where α and β are real. Let us also write D (W) for the fraction of W electrons

penetrating the boundary peak and emerging under the influence of the external field F. Then by solving (13) and (14) and using (15) and (16) we find

$$D(W) = \frac{|a|^2 - |a'|^2}{|a|^2} = \frac{4\beta\left(\frac{C-W}{F}\right)^{\frac{1}{2}}\sqrt{F}}{\left\{W^{\frac{1}{4}}\left(\frac{C-W}{F}\right)^{\frac{1}{4}} + \frac{C-W}{\sqrt{F}}\frac{\beta}{W^{\frac{1}{4}}}\right\}^2 + \frac{1}{\kappa^2 W^{\frac{1}{2}}}\left\{\frac{1}{2}\left(\frac{C-W}{F}\right)^{-\frac{1}{4}} + \frac{C-W}{\sqrt{F}}\kappa\alpha\right\}^2}.$$

It remains to calculate α and β. It is easily verified that

$$\beta = \frac{\sqrt{3}}{2}\frac{(I_{\frac{1}{3}}' I_{-\frac{1}{3}} - I_{-\frac{1}{3}}' I_{\frac{1}{3}})}{(I_{-\frac{1}{3}} + \frac{1}{2}I_{\frac{1}{3}})^2 + \frac{3}{4}I_{\frac{1}{3}}^2}.$$

The numerator is the Wronskian of Bessel's equation of purely imaginary argument and we have exactly

$$I_{\frac{1}{3}} I_{-\frac{1}{3}}' - I_{-\frac{1}{3}} I_{\frac{1}{3}}' = -\frac{2\sin\frac{1}{3}\pi}{\pi Q}.$$

For the denominator we can use the asymptotic expansion, Q being large. The denominator is

$$|\sin\tfrac{1}{3}\pi H_{\frac{1}{3}}^{(2)}(e^{-\frac{1}{2}\pi i}Q)|^2 \sim \tfrac{3}{4}\frac{2}{\pi Q}e^{2Q},$$

so that

$$\beta \sim e^{-2Q}.$$

In evaluating α we can use asymptotic values throughout and find $\alpha = 1$. Thus with sufficient accuracy

$$D(W) = \frac{4\left(\frac{C-W}{F}\right)^{\frac{1}{2}}\sqrt{F}e^{-2Q}}{W^{\frac{1}{4}}\left(\frac{C-W}{F}\right) + \frac{1}{\kappa^2 W^{\frac{1}{2}}}\left\{\frac{1}{2}\left(\frac{C-W}{F}\right)^{-\frac{1}{4}} + \frac{C-W}{\sqrt{F}}\kappa\right\}^2}. \qquad (17)$$

By considering the relative numerical order of the various terms in the denominator it is at once found that those independent of κ are dominant. Therefore we find with sufficient accuracy on inserting the value of Q

$$D(W) = \frac{4\{W(C-W)\}^{\frac{1}{2}}}{C}e^{-4\kappa(C-W)^{\frac{3}{2}}/3F}. \qquad (18)$$

§ 3. *The Complete Electron Emission from a Cold Metal.*—The number of electrons N (W) incident on a surface of unit area per unit time with a kinetic energy W normal to the surface has been evaluated by Nordheim* according to Sommerfeld's theory. He finds

$$N(W) = \frac{4\pi mkT}{h^3}L\left(\frac{W-\mu}{kT}\right), \qquad (19)$$

where

$$L(\beta) = \int_0^\infty \frac{dy}{e^{\beta+y}+1}, \qquad (20)$$

and μ is the usual parameter of the electron distribution in the Fermi Dirac statistics equivalent to the thermodynamic partial potential of an electron. Hence the current I is given quite generally by

$$I = \frac{4\pi m\varepsilon kT}{h^3}\int_0^\infty D(W)\,L\left(\frac{W-\mu}{kT}\right)dW, \qquad (21)$$

where ε is the electronic charge. For all low and ordinary temperatures the integrand of $L\{(W-\mu)/kT\}$ is only sensible when $(W-\mu)/kT + y < 0$ and is then unity. Thus at such temperatures a sufficient approximation to $kT\,L\{(W-\mu)/kT\}$ is $\mu - W$ when $W < \mu$ and otherwise zero. The reason for the sufficiency of this approximation is that in actual cases W and μ are in general very large compared with kT. Since μ is considerably less than C, we may then use (18) for D (W) and find

$$I = \frac{16\pi m\varepsilon}{Ch^3}\int_0^\mu W^{\frac{1}{2}}(C-W)^{\frac{1}{2}}(\mu - W)\,e^{-4\kappa(C-W)^{\frac{3}{2}}/3F}dW.$$

Since the exponent in the integrand is still very large for the largest values of W, it is easy to evaluate this integral to a sufficient approximation. We find, using (6) and putting $C - \mu = \chi$, that

$$I = \frac{\varepsilon}{2\pi h}\frac{\mu^{\frac{1}{2}}}{(\chi+\mu)\chi^{\frac{1}{2}}}F^2 c^{-4\kappa\chi^{\frac{3}{2}}/3F}. \qquad (21)$$

The χ of this equation is necessarily and exactly the thermionic work function.†

If, for convenience of discussion, we express I in amperes per square centimetre of emitting surface, μ and χ in volts and F in volts per centimetre, and insert numerical values for the other constants, we find

$$I = 6{\cdot}2 \times 10^{-6}\frac{\mu^{\frac{1}{2}}}{(\chi+\mu)\chi^{\frac{1}{2}}}F^2 e^{-2{\cdot}1\times 10^7\chi^{\frac{3}{2}}/F}. \qquad (22)$$

We are now in a position to compare the theory with the experimental facts. We see at once that I is of the correct form by comparison with (1) or (2). We have not calculated explicitly the temperature effect on (22), but it is easy

* Nordheim, *loc. cit.*, formula (11).

† Fowler, 'Roy. Soc. Proc.,' A, vol. 117, p. 549 (1928).

o see that it is very small at ordinary temperatures. Formula (22) refers strictly to the limit T = 0, but it is a valid approximation so long as, let us say, χ/kT is very large. Now μ is of the order of 5 volts and kT is $8{\cdot}6 \times 10^{-5}T$ in the same units. This is sufficient to guarantee the observed independence of T for all ordinary temperatures.

The form of the exponent $\chi^{\frac{3}{2}}/F$ is interesting, but there seems no possibility at present of any experimental test of the form.

We come now to the absolute value of the exponent. For the metals commonly experimented with we may take $\chi^{\frac{1}{2}} = 10$, and we can take the exponential factor nearly enough to be $10^{-10^7/F}$. This will make the emission begin to be sensible for fields of rather more than 10^7 volts/cm. The emission would be already very large for fields of the order 10^8 volts/cm. These values are higher than those commonly derived from experiments, which are commonly taken to indicate measurable emission for values of F about 10^6 volts/cm. The absolute determination of F is, however, very difficult, owing to the very large effect of minute surface irregularities or peaks near which we shall find larger values of F than those derived from the geometry of the apparatus. There is also the possibility of sensitive spots on the surface with a reduced value of χ. An investigation by Rother* seems to support this view, for he finds that after very careful preliminary heat treatment an F as large as 10^7 volts/cm. is required to extract a reasonable current. It seems fair to conclude that the phenomenon of electron emission in intense fields is yet another phenomenon which can be accounted for in a satisfactory quantitative manner (at least in broad outline) by Sommerfeld's theory.

§ 4. *Additional Calculations.* It is not difficult to show generally, by comparison with Jeffreys'† asymptotic solution of a similar equation, that for values of W not too near C the exact form of the potential peak and the rounding off at the top will not seriously affect the emission coefficient D (W). Our direct calculation for this very simple case is therefore sufficiently typical. For the Q of an exact solution has to be replaced by

$$\kappa \int_{x_0}^{x_1} (V - W)^{\frac{1}{2}}\, dx,$$

where V is the potential energy of the electron at any point and x_0 and x_1 are the points at which V − W vanishes. The integration range is shown in fig. 3. It is at once clear that, provided the shaded area is reasonably large, modifications in the contour near the peak are unimportant.

* Rother, 'Ann. der Physik,' vol. 81, p. 316 (1926).

† Jeffreys, 'Proc. Lon. Math. Soc.' vol. 23, p. 428 (1924).

We pass on to an outline of the calculations necessary to establish the form of the combined thermionic and strong field emission. In the first place, if

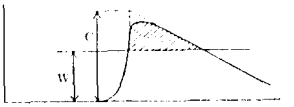


Fig. 3.

W > C, the calculation of D (W) can be carried out exactly as in Nordheim's paper, and we find

$$D(W) = \frac{4W^{\frac{1}{2}}(W - C)^{\frac{1}{2}}}{[W^{\frac{1}{2}} + (W - C)^{\frac{1}{2}}]^2 + [F/(4\kappa\{W - C\})]^2}. \qquad (23)$$

From this formula it is easily verified that the term in F is quite unimportant in ordinary cases, and the emission can be calculated as if F = 0.

Near the peak itself, when W is nearly equal to C, neither formula (18) nor (23) can be strictly used, and D (W) will depend essentially on the exact form of the potential energy curve. All we can say at present is D (W) = 0 when $W = V_{max}$, the maximum ordinate, at which in general the tangent will be horizontal, and that for other neighbouring values of W D (W) is of the order of $\frac{1}{2}$.

We can now break up the range of integration of W in the general formula (21) into four parts as in fig. 4. We can see at once that of the total emission

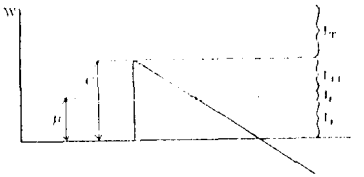


Fig. 4.

I_T and I_F represent fairly accurately the purely thermionic and purely strong field emission respectively. In I_1 (which, however, is unimportant), a more accurate value of L $\{(W - \mu)/kT\}$ must be used than in I_F, and both I_1 and I_{TF} depend essentially on both T and F. Calculations for the field of fig. 1 are not of sufficient importance to give in detail, but they give no combined result of the form (3) nor any grounds for expecting such a result for any natural modification of the field.